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Mathematical Games

Randomization, derandomization and antirandomization: three games

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Abstract

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Three games are given between two players, Paul and Carole, with a common theme. In each round Paul does a split and Carole chooses. Random play by Carole allows a bound for the game value. Through derandomization this becomes a deterministic strategy for Carole minimizing a weight function. Paul can use that same weight function to give a bound for the game value in the other direction.

1. The Tenure game

DICK				
RAVI	FAN			
SHAFI	LACI		DON	
PostD	AP1	AP2	Assoc	Tenure

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The tenure game is a perfect information game between two players, Paul — chairman of the department — and Carole — dean of the school. An initial position is given in which various faculty members (DICK, RAVI, etc.) are at various pretenured positions. Paul will win if some faculty member receives tenure — Carole wins if no faculty member receives tenure. Each year (or round if you will) Chair Paul creates a promotion list L of the faculty[1] and gives it to Dean Carole who has two options. Option One: Carole may promote all faculty on list L one rung and simultaneously fire all other faculty. Option Two: Carole may promote all faculty *not* on list L one rung and simultaneously fire all faculty on list L. With the example above, suppose $L=\{\mathrm{DON}, \mathrm{SHAFI}\}$. If Carole applies Option One DON receives tenure and Paul has won. So Carole would apply Option Two: DON and SHAFI would disappear, FAN and LACI would become level two Assistant Professors and RAVI and DICK would become level one Assistant Professors. The next year Paul presents another list L and Carole picks one of the two options. The Tenure game represents an extreme form of "publish or perish", within four years all faculty will either have been promoted to tenure or fired. With perfect play on both sides, who wins the Tenure game?

Naturally we shall consider a general opening position, let us suppose that there are a_k faculty that are k rungs from tenure and that k can be arbitrarily large, though bounded.

Theorem 1. *If*

$$\sum a_k 2^{-k} < 1,$$

then Carole wins.

First proof. Let us imagine that Carole plays *randomly*, i.e., each round after Paul has determined the promotion list L Carole flips a fair coin to decide whether to use Option one or Option two. Fix some deterministic strategy for Paul. Now each faculty has a probability of reaching tenure — for the example above FAN has probability $\frac{1}{8}=2^{-3}$ of receiving tenure since for the next three years Carole must select the Option that promotes, rather than fires, FAN. Note critically that this probability is 2^{-3} regardless of Paul's strategy; when Paul puts FAN in L Carole must choose Option One while when Paul leaves FAN out of L Carole must choose Option Two but each occurs with probability $\frac{1}{2}$. Let T be the number of faculty receiving tenure so that T is a random variable. For each faculty member f let I_f be the indicator random variable for f receiving tenure so that $T=\sum I_f$. Then by linearity of expectation

$$E[T]=\sum E[I_f]=\sum a_k 2^{-k},$$

as those f which are k rungs from tenure each have $E[I_f]=2^{-k}$. Note that Carole wins if and only if $T=0$. Our assumption is that $E[T]<1$ and hence

$$\Pr[\text{Carole wins}]=\Pr[T=0]>0.$$

[1] The faculty are only pawns in this game!

Now comes the slick part. The Tenure game is a finite perfect information game with no draws so that either Paul or Carole has a perfect strategy. Had Paul had a perfect strategy then by playing it the probability of Carole winning would be zero, which is not the case. Hence, Carole *must* have a winning strategy! □

The above proof is a nice example of the probabilistic method, the use of probabilistic analysis to prove a deterministic result. As often the case with the probabilistic method it leaves open the question of actually finding the desired object — in this case Carole's strategy. The "removal of the coin flip" to give a deterministic object is generally called *derandomization*.

Second proof (Derandomization). Define the weight of a position as the expected number $E[T]$ of faculty receiving tenure if Carole plays randomly. Explicitly, with a_k faculty k rungs from tenure the weight is $\sum a_k 2^{-k}$. Now Paul presents a list L to Carole. Let T^1 be the number of faculty receiving tenure if Carole now plays Option One and then plays randomly in all succeeding rounds. Let T^2 be the same with Carole first playing Option Two. Carole's strategy is to pick Option One if $E[T^1] < E[T^2]$, otherwise to pick Option Two. (Suppose Option One leaves b_k players k rungs from tenture after its application while Option Two leave c_k players k rungs from tenure. Then Carole simply checks if $\sum b_k 2^{-k} < \sum c_k 2^{-k}$ and hence this is a very efficient strategy.) The key point here is that

$$E[T] = \tfrac{1}{2}(E[T^1] + E[T^2]),$$

since playing randomly throughout is the average of playing Option One and then randomly and playing Option Two and then randomly. As $E[T] < 1$ either $E[T^1] < 1$ or $E[T^2] < 1$ and employing this strategy Carole ensures that $E[T] < 1$ at the end of the round. But at the end of the game $E[T]$ is simply the number of faculty who have received tenure. An integer less than one must be zero so Carole has won. □

The Tenure game has the nice property that when the condition for Carole winning does not hold Paul can use this same weight function to give a winning strategy for himself. We coin the term *antirandomization* to describe this process. We need in this case an amusing lemma.

Splitting lemma. *Let $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_r$ all be negative powers of two with sum $x_1 + \cdots + x_r = 1$. Then there exists a partition of the x_i into two groups so that each group sums to at precisely one half.*

Proof. We place the x_i into groups largest first, always placing x_i into the group with the currently smaller sum. Let us say we are stuck at l if after placing $x_1, \ldots, x_l$ the difference of the sums of the groups (in absolute value) is greater than the sum $x_{l+1} + \cdots + x_r$ of the as yet unplaced x's. We show by induction on l, $0 \leqslant l \leqslant r$, that we are never stuck. We are trivially not stuck at $l=0$, assume by induction that we are not stuck at $l-1$. Case 1: the two groups currently have different sums. As all $x_1, \ldots, x_{l-1}$ are multiples of x_l the difference of the sums of the groups must be a multiple of x_l. Hence the difference is at least x_l and so placing x_l in the smaller group cannot make us stuck. Case 2: the two groups currently have the same sum. This sum, as in Case 1, must be of the form Ax_l, A integral. Thus $x_1+\cdots+x_l$ is of the form $(2A+1)x_l$ and hence

$$x_{l+1}+\cdots+x_r=1-(2A+1)x_l \geqslant x_l,$$

so that after placing x_l in either group we are not stuck. Hence we will not be stuck at $l=r$ which means that after placement of all $x_1, \ldots, x_l$ the sums are precisely the same. □

Corollary. *Let $x_1 \geqslant \cdots \geqslant x_l$ be negative powers of two with sum at least one. Then there is a partition of the x_i into two groups so that each group sums to at least one half.*

Proof. If $x_1+\cdots+x_l>1$ then, since it is a multiple of x_l, $x_1+\cdots+x_{l-1} \geqslant 1$. Remove x_l, $x_{l-1}, \ldots$ until $x_1+\cdots+x_r=1$ and apply the Splitting lemma. □

Theorem 2. *If*

$$\sum a_k 2^{-k} \geqslant 1,$$

then Paul wins the Tenure game.

Proof. Initially $E[T] \geqslant 1$. From the Splitting lemma Paul may create a list L so that $E[T^1] \geqslant 1$ and $E[T^2] \geqslant 1$. (Note that $E[T^1]$ is defined after Carole plays Option One and so is double the sum of the original weights of the faculty in list L.) Regardless of what Carole does $E[T] \geqslant 1$ at the end of the round. At the end of the game $E[T] \geqslant 1$ and thus someone has received tenure and Paul has won. □

The Splitting lemma enabled us to give a precise solution to the Tenure game. In future examples we will not be so fortunate but the notions of randomization, derandomization and antirandomization will remain.

2. The balancing vector game

This is a perfect information game between two players, again Paul and Carole, with parameter n. There is a position vector $P \in \mathbb{R}^n$ which is originally set to 0. There are n rounds. (The more general situation in which the number of rounds and the dimension are two separate parameters is also interesting but we do not discuss it here.) On each round Paul first selects a vector $v \in \{-1, +1\}^n$. Carole then resets P to either $P+v$ or $P-v$, her choice. Let P^{final} denote the value of P at the end of the game.

The payoff to Paul (from Carole) is then $|P^{\text{final}}|_\infty$, i.e., the largest absolute value of the n coordinates of P^{final}.

As a finite perfect information zero-sum game there is a value, which we will denote $VAL(n)$. It will be convenient to define for $\alpha \geqslant 0$ the (α, n)-game: Paul wins the (α, n) game if and only if $|P^{\text{final}}|_\infty \geqslant \alpha$. Note that Paul wins the (α, n) game if and only if $VAL(n) \geqslant \alpha$ so that determination of the winner of the (α, n) game for various α will give bounds on $VAL(n)$.

Notation. S_n is the random variable with distribution

$$S_n = X_1 + \cdots + X_n,$$

where $\Pr[X_i = +1] = \Pr[X_i = -1] = \frac{1}{2}$ and the X_i are mutually independent.

Theorem 3. *If*

$$n \Pr[|S_n| \geqslant \alpha] < 1,$$

then Carole wins the (α, n) *Balancing Vector game.*

First proof. Let us imagine that Carole plays *randomly*, i.e., each round after Paul has determined $v \in \mathbb{R}^n$ Carole flips a fair coin to decide whether to change P to $P+v$ or $P-v$. Fix some deterministic strategy for Paul. Now each coordinate has a probability of having absolute value at least α at the end of the game. Note critically that this probability is $\Pr[|S_n| > \alpha]$ regardless of Paul's strategy; Paul can make the coordinate in v either $+1$ or -1 but either way the coordinate of P is changed by $+1$ or -1 with probability $\frac{1}{2}$. Thus the coordinate in P^{final} has distribution S_n. Let T be the number of coordinates with absolute value at least α in P^{final} so that T is a random variable. For each coordinate $1 \leqslant i \leqslant n$ let I_i be the indicator random variable for the ith coordinate having absolute value at least α in P^{final} so that $T = \sum I_i$. Then by Linearity of Expectation

$$E[T] = \sum_{i=1}^{n} E[I_i] = n \Pr[|S_n| \geqslant \alpha].$$

Note that Paul wins if and only if $T \geqslant 1$. Our assumption is that $E[T] < 1$ so that

$$\Pr[\text{Paul wins}] = \Pr[T \geqslant 1] < 1.$$

The (α, n) game is a perfect information game with no draws so that either Paul or Carole has a perfect strategy. Had Paul had a perfect strategy he could win with probability one, which is not the case. Hence, Carole must have a perfect strategy. □

As with the Tenure game we use this randomized strategy to yield a weight function which gives a deterministic strategy.

Second proof (Derandomization). Define the weight of a position to be the expected number $E[T]$ of coordinates of P^{final} with absolute value at least α if Carole plays randomly for the remainder of the game. Explicitly, suppose $P=(x_1,\ldots,x_n)$ and there are r rounds remaining in the game. Then P has weight

$$w(P)=\sum_{i=1}^{n} \Pr[|x_i+S_r|\geqslant\alpha].$$

(Formally, the weight is a function of P and r.) At a position P Paul now presents $v\in\{-1,+1\}^n$ to Carole. Carole's strategy is to change P to either $P+v$ or $P-v$, whichever has the smaller weight. (The weights are sums of binomial coefficients and so may be calculated efficiently. Each round Carole needs calculate only two such weights.) The key point here is that

$$w(P)=\tfrac{1}{2}(w(P+v)+w(P-v)),$$

since playing randomly throughout is the average of playing $P\leftarrow P+v$ and then randomly and playing $P\leftarrow P-v$ and then randomly. The original $P=0$ has $w(P)=n\Pr[|S_n|\geqslant\alpha]<1$ by assumption. With $w(P)<1$ either $w(P+v)<1$ or $w(P-v)<1$ so with this strategy Carole ensures that the new $w(P)<1$. Continuing this at the end of the game $w(P)<1$. But at the end of the game $w(P^{\text{final}})$ is simply the number of coordinates with absolute value at least α. An integer less than one must be zero and so Carole has won. $\square$

Now we want to apply antirandomization to give a strategy for Paul. The precise Splitting lemma of the Tenure game cannot be duplicated in the context of the Balancing Vector game but we can give an approximate splitting Lemma. Let $P=(a_1,\ldots,a_n)$ be the position vector with $r+1$ rounds remaining. Suppose Paul then plays $v=(\varepsilon_1,\ldots,\varepsilon_n)$. Then

$$w(P+v)-w(P-v)=\sum_{i=1}^{n} \Pr[|a_i+\varepsilon_i+S_r|\geqslant\alpha]-\Pr[|a_i-\varepsilon_i+S_r|\geqslant\alpha].$$

The effect of flipping ε_i from $+1$ to -1 is to reverse its effect on $w(P+v)-w(P-v)$ so that

$$w(P+v)-w(P-v)=\sum_{i=1}^{n} \varepsilon_i z_i,$$

where we set

$$z_i=\Pr[|a_i+1+S_r|\geqslant\alpha]-\Pr[|a_i-1+S_r|\geqslant\alpha].$$

Observe that we can now write

$$z_i=\Pr[S_r=w]-\Pr[S_r=w'],$$

where w is the unique integer of the same parity of r such that $a_i+1+w\geqslant\alpha$ but $a_i-1+w<\alpha$ and w' is the unique integer of the same parity of r such that

$a_i - 1 + w' \leqslant -\alpha$ but $a_i + 1 + w' > -\alpha$. (Note any value of S_r must have the parity of r.) We may therefore bound

$$|z_i| \leqslant \max_w \Pr[S_r = w] = \binom{r}{\lfloor r/2 \rfloor} 2^{-r}.$$

Approximate Splitting lemma. *With $r+1$ moves remaining in the Balancing Vector game Paul can select $v = (\varepsilon_1, \ldots, \varepsilon_n)$ such that*

$$|w(P+v) - w(P-v)| \leqslant \binom{r}{\lfloor r/2 \rfloor} 2^{-r}.$$

Proof. Select ε_i sequentially always minimizing the absolute value of the partial sum $\varepsilon_1 z_1 + \cdots + \varepsilon_l z_l$. With any bound K on $|z_i|$ this greedy algorithm assures that all such absolute values will be at most K. □

Theorem 4. *If*

$$n \Pr[|S_n| \geqslant \alpha] > \frac{1}{2} \sum_{r=0}^{n-1} \binom{r}{\lfloor r/2 \rfloor} 2^{-r},$$

then Paul wins the (α, n) Balancing Vector game.

Proof. Paul's strategy is to select v such that $w(P+v)$, $w(P-v)$ are as close together as possible. As $w(P)$ is always the average of $w(P+v)$, $w(P-v)$ the v of the Approximate Splitting lemma assures that

$$w(P \pm v) \geqslant w(P) - \frac{1}{2} \binom{r}{\lfloor r/2 \rfloor} 2^{-r},$$

when there are $r+1$ rounds remaining. Initially $w(0) = n \Pr[|S_n| \geqslant \alpha]$ so at the end of the game $w(P)$ is still positive. But $w(P^{\text{final}})$ is the number of coordinates of absolute value at least α and when this is positive Paul has won. □

The asymptotics of $VAL(n)$ are found from the above theorems by using the approximation

$$\Pr[|S_n| > \alpha] = e^{-(\alpha^2/2n)(1+o(1))},$$

which is valid (we omit details) when $\alpha n^{-1/2} \to \infty$ and $\alpha = n^{1/2 + o(1)}$. We also use the approximation

$$\sum_{r=0}^{n-1} \binom{r}{\lfloor r/2 \rfloor} 2^{-r} = \Theta\left(\sum_{r=0}^{n-1} (1+r)^{-1/2} \right) = \Theta(n^{1/2}).$$

Together these results yield the following theorem.

Theorem 5.

$$\sqrt{n\ln n}\,(1+o(1))<VAL(n)<\sqrt{2n\ln n}\,(1+o(1)).$$

Finding the correct constant in the asymptotic evaluation of $VAL(n)$ remains a vexing question.

3. Paul and Carole games

The games we are considering have a common theme. In all cases Paul each round makes a play and then Carole can either accept the play or do its opposite.

- *Randomization.* We first analyze a random strategy for Carole. When we can show that this strategy wins with positive probability this implies (as the games are all perfect information with no draws) that Carole can always win.
- *Derandomization.* We define the weight function of a position as the expected number of bad things that will happen (and cause Carole to lose) if Carole were to play randomly. Now we create a deterministic strategy for Carole by having her always play so as to minimize this weight function.
- *Antirandomization.* Paul now uses *this* weight function for effective counterplay. For any move by Paul the average of the weights of the potential new positions is the weight of the old position. Paul now plays so as to make these two potential new weights as close together as possible. Then Carole cannot lower the weight very much and so if the initial weight was sufficiently high it must end up greater than zero and Carole has lost.

The games have been motivated partially by consideration of on-line algorithms — the Balancing Vector game being the best example. Here Paul is going to receive n vectors $v_1,\dots,v_n\in\{-1,+1\}^n$ and wants to choose $\varepsilon_1,\dots,\varepsilon_n\in\{-1,+1\}$ so that the signed sum $\varepsilon_1 v_1+\cdots+\varepsilon_n v_n$ is small. Indeed this author [3] has shown that there exists a choice of ε_i so that this signed sum has L^∞-norm $O(\sqrt{n})$. Here, however, Paul requires an *online* algorithm that determines ε_i immediately upon seeing v_i. Carole is an adversary and her strategy shows that in the worst case analysis Paul cannot keep the L^∞-norm lower that $\Theta(\sqrt{n\ln n})$.

These approaches have been used in the recent book [1]. The derandomization approach, sometimes called the method of conditional expectations, can be found in Raghavan [2].

The specific names Paul and Carole were not randomly chosen. The initials P and C refer to Pusher–Chooser games investigated by this author. Paul may be considered the great questioner, Paul Erdős. And Carole may be thought of by her acronym — Oracle!

4. The Liar game

Here we begin, as with the Tenure game, with a_i chips on square i for $0\leqslant i\leqslant k$. The number of rounds is specified in advance and denoted by q. On each round Paul selects a set L of chips. In this game Carole again has two options. Option One: move all chips in L up one square. Option Two:move all chips not in L up one square. (Unlike the Tenure game the "other" chips remain on the board.) Chips that are moved forward from square k are eliminated from the board. Paul wins if after the q rounds there is at most one chip remaining on the board.

In vector format when there are b_i chips on square i we call $P=(b_0,\dots,b_k)$ the position vector. When the set L has c_i chips on square i we call $v=(c_0,\dots,c_k)$ Paul's move vector. We define $P+{}^*v$ and $P-{}^*v$ to be the new position vectors if Carole plays Options One or Two, respectively. Explicitly

$$P+{}^*v=(b_0-c_0, b_1-c_1+c_0, \dots, b_i-c_i+c_{i-1}, \dots b_k-c_k+c_{k-1}),$$

$$P-{}^*v=(c_0, c_1+b_0\div c_0, \dots, c_i+b_{i-1}-c_{i-1}, \dots c_k+b_{k-1}-c_{k-1}).$$

The above is a chip formulation of the following Liar game. Let A_i be disjoint sets of size a_i, $0\leqslant i\leqslant k$ and let Ω be their union. Suppose Paul is trying to find an unknown $x\in\Omega$ by asking q questions of Carole, all of the form "Is $x\in L$?" When $x\in A_i$ we allow Carole to lie at most $k-i$ times. (We may think of this is an intermediate stage of a game in which initially Carole was allowed to lie at most k times but where A_i is those x for which if x is the answer Carole has already lied i times.) This becomes a perfect information game by allowing Carole to play an adversary strategy of not actually picking an x beforehand but rather answering in a way consistent with at least one x. A "No" answer by Carole corresponds to moving all chips in L up one square while a "Yes" answer corresponds to moving all chips not in L up one square. The chips remaining at the end of q rounds correspond to possible values x. When no chips remain Carole has cheated but we adjust the rules by allowing her to cheat, insisting that if she cheated she has lost. With this modification Paul wins if there is *at most* one chip remaining at the end of the game. Here we will concentrate on the chip version. The results of this section have been given in [4] though the proofs given here (especially for Paul's strategy) are somewhat different.

Let $B(s,\frac{1}{2})$ denote the usual binomial distribution, the number of heads in s independent flips of a fair coin.

Theorem 6. *If*

$$\sum_{i=0}^{k} a_i \Pr[i+B(q,\tfrac{1}{2})\leqslant k]>1,$$

then Carole wins the Liar Game.

First proof. Let us imagine that Carole plays *randomly*, i.e., each round after Paul has determined the set L Carole flips a fair coin to decide whether to use Option One or Option Two. Fix some deterministic strategy for Paul. Now each chip has a probability of remaining on the board. Note critically that, for a chip initially on square i, this probability is $\Pr[i+B(q,\frac{1}{2})\leqslant k]$ regardless of Paul's strategy. This is because on each round whether Paul places the chip in L or not it has probability $\frac{1}{2}$ of moving forward one square. Let T be the number of chips remaining on the board. Then $T=\sum I_x$ where I_x is the indicator random variable for chip x remaining on the board and the sum is over all chips. Then by linearity of expectation

$$E[T]=\sum E[I_x]=\sum_{i=0}^{k} a_i \Pr[i+B(q,\tfrac{1}{2})\leqslant k].$$

Note that Paul wins if and only if $T\leqslant 1$. Our assumption is that $E[T]>1$ so that

$$\Pr[\text{Paul wins}]=\Pr[T\leqslant 1]<1.$$

The Liar game is a perfect information game with no draws so that either Paul or Carole has a perfect strategy. Had Paul had a perfect strategy he could win with probability one, which is not the case. Hence, Carole must have a perfect strategy. □

Second proof (Derandomization). Define the weight of a position to be the expected number $E[T]$ of chips at the end of the game if Carole plays randomly for the remainder of the game. Explicitly, the position $P=(b_0,\ldots,b_k)$ with r moves remaining has weight

$$w(P)=\sum_{i=0}^{k} b_i \Pr[i+B(r,\tfrac{1}{2})\leqslant k].$$

(Again, the weight is formally a function of P and r.) At a position P Paul now presents a move vector v to Carole. Carole's strategy is then to play that option which gives the new position ($P+{}^*v$ or $P-{}^*v$) with the highest weight. (The weights are sums of binomial coefficients and so may be calculated efficiently. Each round Carole needs calculate only two such weights.) The key point here is that

$$w(P)=\tfrac{1}{2}(w(P+{}^*v)+w(P-{}^*v))$$

since playing randomly throughout is the average of playing Option One and then randomly and playing Option Two and then randomly. The original position had

$$w(P)=\sum_{i=0}^{k} a_i \Pr[i+B(q,\tfrac{1}{2})\leqslant k]>1,$$

by assumption. Carole's strategy assures that the weight does not decrease so that the final weight is greater than one. But the final weight is the number of chips remaining and so Carole has won. □

Now we want to apply antirandomization to give a strategy for Paul. We call a move vector v a perfect split if $w(P+{}^*v)=w(P-{}^*v)$. The precise Splitting lemma of the tenure game *can* be duplicated in the context of the liar game but only with some additional assumptions on the position. Complicating matters, we need a Splitting lemma that allows Paul to continue making perfect splits throughout the q moves of the game.

Splitting lemma. *Let $P=(b_0,\dots,b_k)$ be a position vector with $r+1$ moves remaining with $w(P)=1$. Assume further that*

$$b_k \geqslant \binom{r}{k},$$

and that $r \geqslant 2k$. Then there exists a perfect split $v=(c_0,\dots,c_k)$ such that

$$c_i = \left\lfloor \frac{b_i}{2} \right\rfloor \quad \text{or} \quad c_i = \left\lceil \frac{b_i}{2} \right\rceil \ \text{for } 0 \leqslant i < k,$$

and

$$\left| c_k - \frac{b_k}{2} \right| < \frac{1}{2}\binom{r}{k}.$$

Proof. It will be convenient to define

$$\Delta = w(P - {}^*v) - w(P + {}^*v)$$

Consider the effect on Δ or placing a single chip, initially on square i, on Paul's list L. With Option One the chip goes to square $i+1$ with weight $\Pr[i+1+B(r,\frac{1}{2}) \leqslant k]$; with Option Two it stays on square i with weight $\Pr[i+B(r,\frac{1}{2}) \leqslant k]$ and so the difference is $\Pr[i+B(r,\frac{1}{2})=k]$, or $\binom{r}{k-i}2^{-r}$. If the chip is left off L the effects are reversed and Δ is decreased by the same amount. When there are an even number b_i chips on square i placing $c_i = b_i/2$ of them on the list L has zero effect on Δ. For those $0 \leqslant i < k$ were b_i is odd Paul alternately places the odd chip in or out of L, splitting the remaining chips in half. The effect on Δ is then an alternating series of terms. With $r \geqslant 2k$ the values $\binom{r}{k-i}$ decrease for $0 \leqslant i < k$ so these terms are decreasing in absolute value. It will be convenient to call the chips on square k *pennies* and the other *nonpennies*. After placing all the nonpennies the absolute value of Δ is at most $\binom{r}{k}2^{-k}$ and is of the form $a2^{-r}$ for some integer a. Paul now takes the first a pennies (as $a \leqslant \binom{r}{k} \leqslant b_k$ by assumption) and places them either all in or all out of L so as to make $\Delta = 0$. Now if there are an even number of pennies remaining Paul splits them evenly and the splitting is complete. But we claim that must be the case. If not Paul could split them except for one penny and therefore make the final $\Delta = 2^{-r}$. As $w(P-{}^*v)+w(P+{}^*v)=w(P)=1$ this would give $w(P-{}^*v)=(1+2^{-r})/2$ but all weights with r moves remaining are clearly multiples of 2^{-r}, a contradiction. □

Theorem 7. *For every k there exists $q_0 = q_0(k)$ so that for $q > q_0$ the following holds for the q move Liar game: If $P=(x_0,\dots,x_k)$ is an initial position vector with weight one and*

$$x_k \geqslant 2\binom{q-1}{k} + 2^2\binom{q-2}{k} + \cdots + 2^k\binom{q-k}{k},$$

then Paul wins the Liar game.

Proof. Paul applies the strategy of the Splitting lemma repeatedly. For the first k rounds the bound on the initial x_k combined with the near-halving given by the Splitting lemma assure that the number of pennies remains adequate. Let $c=c(k)$ be a large constant to be chosen shortly. We now show by induction on j from $q-k$ to c that when there are j rounds remaining the number of pennies is at least $\binom{j}{k}$. This holds for $j=q-k$ by the choice of the initial x_k. Assume by induction that it held for $j'>j$ so that in particular with $j+k$ moves remaining there was a position vector P with (as all splits were perfect) $w(P)=1$. Now we do some rough asymptotics in j for fixed k. The maximal weight of a chip with $j+k$ moves remaining was $\binom{j+k}{k}2^{-j-k}$ so there were $\Omega(2^j j^{-k})$ chips and hence $\Omega(2^j j^{-k})$ chips on some particular square. If they were on a square $i<k$ then for i rounds at least floor of half of them remained where they were and then for $k-i$ rounds at least floor of half of them moved forward one square so that with j rounds to go a positive proportion of them, $\Omega(2^j j^{-k})$, are pennies. If $\Omega(2^j j^{-k})$ were all pennies then each round the number of pennies is at least half minus $\binom{j+k}{k}$ what it was before so with j rounds remaining one still has $\Omega(2^j j^{-k})-O(j^k)=\Omega(2^j j^{-k})$ pennies. We fix c with $c\geqslant 2k$ so that for $j\geqslant c$ these two expressions are both at least $\binom{j}{k}$.

Note c depends only on k, not on q. Now we want to show that for q sufficiently large the position with c moves remaining will be particularly simple. Define a new weight $w^*(P)$ of a position P to be the expected number of *nonpennies* that will remain when there are c moves remaining in the game if Carole plays randomly. With $P=(x_0,\dots,x_k)$ and r moves remaining this means

$$w^*(P)=\sum_{i=0}^{k-1} x_i \Pr[i+B(r-c,\tfrac{1}{2})\leqslant k-1].$$

For, i, k, c fixed asymptoticlly in q we note that

$$\Pr[i+B(q,\tfrac{1}{2})\leqslant k]=\Theta(q^{k-i}2^{-q}),$$
$$\Pr[i+B(q-c,\tfrac{1}{2})\leqslant k-1]=\Theta(q^{k-i-1}2^{-q}),$$

so that the second will be smaller for q sufficiently large. With c, k already fixed we let q_0 be such that for $q\geqslant q_0$ and every $0\leqslant i<k$ the second is smaller.

For $q\geqslant r\geqslant c$ let $P^{(r)}$ denote the position with r moves remaining. Our choice of q_0 assures that initially

$$w^*(P^{(q)})<w(P^{(q)})=1.$$

For $q>r\geqslant c$ we define

$$\Delta^{(r)}=w^*(P^{(r)})-w^*(P^{(r+1)}).$$

As in the splitting lemma every even pile of chips cancels out and $\Delta^{(r)}$ is an alternate sum of the effects of a single chip. As $c\geqslant 2k$ the largest possible term would come from a chip on square zero so that we may bound

$$\Delta^{(r)}\leqslant \Pr[B(r-c,\tfrac{1}{2})\leqslant k-1]-\Pr[B(r+1-c,\tfrac{1}{2})\leqslant k-1],$$

so that $\Delta^{(q-1)}+\cdots+\Delta^{(c)}$ is bounded by a telescoping sum which is at most one. Then $w^*(P^{(c)})$ is at most one more that $w^*(P^{(q)})$, hence is less than two. But $w^*(P^{(c)})$ is the number of nonpennies with c moves remaining. That is, with c moves remaining there are either no nonpennies or precisely one nonpenny.

Paul now employs a simple endgame strategy for the final c moves. Let $0 \leqslant d < c$ and suppose with $d+1$ moves remaining there are no nonpennies. Then Paul simply splits the pennies in half. Otherwise there is precisely one nonpenny and some a pennies. Let $f(x)$ be what the weight would become if Paul selects the nonpenny and x pennies and the Carole chooses Option One. Then $f(0) \leqslant 1$ since only one chip would remain on the board. $f(a) \geqslant 1$ since all the chips would remain where they were but their weights would not decrease. For any x the difference $f(x+1)-f(x)=2^{-d}$, the new weight of a penny. As $f(0)$, $f(a)$ are multiples of 2^{-d} there will be an x with $f(x)=1$ and Paul makes this split.

Paul has managed to find perfect splits for the entire game so that with no moves remaining the weight is still one and therefore precisely one chip remains and Paul has won. □

The natural opening situation for the Liar game is when $a_0=n$ and $a_i=0$ for $1 \leqslant i \leqslant k$. Of course, the conditions of the above theorems do not apply in this case. For k fixed and q sufficiently large necessary and sufficient conditions on n are found in [4] for Paul to win with this opening position. This result is essentially a corollary of the above theorems.

5. The Tenure game revisited and reversed

We first generalize the goal of the Tenure game — let the payoff to Paul be the *number* of faculty receiving tenure. That is, Paul is trying to maximize the number of faculty receiving tenure while Carole is trying to minimize this number. As with the Balancing Vector game we define the α-Tenure game to be a win for Paul if at least α faculty receive tenure.

Theorem 8. *Let $w=\sum a_k 2^{-k}$ be the weight of the initial position in the generalized Tenure gamehen the value V of the game (to Paul) is $\lfloor w \rfloor$.*

Proof. Let a be an integer with $w<a$ and consider the a-Tenure game. When Carole plays randomly any strategy of Paul gives an expected number w faculty receiving tenure so that the probability of Paul winning is less than one and therefore Carole must have a winning strategy. Hence $V<a$.

Let a be an integer with $w \geqslant a$ and consider the a-Tenure game. A straightforward generalization of the Splitting lemma is that if $x_1 \geqslant \cdots \geqslant x_l$ are negative powers of two with sum at least a then there is a partition of the x_i into two groups so that each group has sum at least $a/2$. Applying this Paul can repeatedly assure that the weight is at least a and at the end of the game the weight is the number of faculty that have received tenure. Thus $V \geqslant a$. But these together imply $V = \lfloor w \rfloor$. $\square$

Now we *reverse* the Tenure game. The rules are the same and the payoff is the number of faculty receiving tenure but now the payoff is to Carole. That is, Paul is trying to minimize the number of faculty receiving tenure while Carole is trying the maximize that number. Let V^r denote the value of this reversed game. We define the α-reversed Tenure game to be a win for Carole if at least α faculty receive tenure. It is perhaps surprising that the analysis of the reversed game is quite similar to the original game.

Theorem 9. *Let* $w = \sum a_k 2^{-k}$ *be the weight of the initial position in the generalized reversed Tenure game. Then the value* V^r *of the game (to Carole) is* $\lceil w \rceil$.

Proof. Let a be an integer with $w \geqslant a$ and consider the a-Tenure game. When Carole plays randomly any strategy of Paul gives an expected number w faculty receiving tenure so that the probability of Paul winning is less than one and therefore Carole must have a winning strategy. Hence $V^r \geqslant a$.

Let $a = \lceil w \rceil$ and consider the a-Tenure game. We need the following reversal of the Splitting lemma: Let $x_1, \ldots, x_l$ be negative powers of two with sum at most a. Then there is a partition of the x_i into two groups each of sum at most $a/2$. To show this add on $x_{l+1}, \ldots$ until the sum is precisely a and then apply the previous Splitting lemma. Now applying this Paul can repeatedly assure that the weight is at most a and at the end of the game the weight is the number of faculty that have received tenure. Thus $V^r \leqslant a$, and hence $V^r = \lceil w \rceil$. $\square$

6. More reversals

The reversing of the objects of Paul and Carole can be applied to the other games as well. For the Balancing Vector game it makes most sense when Carole is trying to maximize the number of coordinates of absolute value at least α, as she may trivially make one coordinate equal to n. These Paul–Carole games have a "Paul splits, Carole chooses" nature. In some games Carole will take the "bigger piece", in the reversals she will take the "smaller piece", but in either case Paul's best strategy is to make as even a split as possible. (The notion of size is given by the weight function.) With that strategy of Paul's, Carole's role is limited and therefore the end result of the game and its reversal tend to be quite similar.

The reversal of the Liar game is particularly intriguing. Lets call it the Prediction game: it has the same rules as the Liar game except that if at the end of the game there is at most chip remaining then Carole is the winner instead of the loser. Surprisingly, this game has come up independently in examination of Abstract Prediction in a study (in preparation) of on line learning by Helbold, Warmuth and Cesa–Bianchi. We let

w be the same weight function as before. Now Carole can play to minimize the weight of a position. If the initial weight is less than two then Carole can assure that the final weight (the number of chips remaining) is less than two, hence at most one and she has won. Freund et al. have examined partial converses of this result analogous to the theorems for the Liar game.

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